Ergodic Theory and Measured Group Theory
Lecture 19

We will focus on part (b) of the last remark, but before doing so, let's explore the possibility of a weaker classifiability of $T \in \mathrm{Aut}(\mu)$ up to $\sim$ than concrete classifiability.

Def. Let E, F be eq. rels on st. Borel spaces X and Y, resp. We call a function $\pi: X \to Y$ a *reduction* from E to F if $\forall x_1, x_2 \in X$, $x_1 E x_2 \iff \pi(x_1) F \pi(x_2)$. We say that E is *Borel reducible* to F, denoted $E \leq_B F$, if $\exists$ a Borel reduction from E to F.

A good next (after $=$) candidate for F is isomorphism between ctbl structures, e.g. graphs, groups, rings. We can encode such structures (in a fixed first-order language) into a Polish space. For example, for ctbl graphs, we can assume their vertex set is $\mathbb{N}$, then the edge set would be a subset of $\mathbb{N}^2$, i.e. an element of $\mathcal{P}(\mathbb{N}^2) \cong 2^{(\mathbb{N}^2)}$. Thus any ctbly infinite graph is an element of $2^{(\mathbb{N}^2)}$ (compact Polish). It's a theorem that the eq. rel. of isomorphism of graphs is an analytic set but it isn't Borel. This good for reducing $\sim$ on $\mathrm{Aut}(\mu)$ to this isom. rel. because if $E \leq_B F$ and F is Borel, then so is E (as before with $=$, E is a Borel preimage of F).

<u>Def.</u> We say that an eq. rel. E on a st. Borel X is **classifiable by ctbl structures** if $\exists$ first-order ctbl language $\mathcal{L}$ s.t. E is Borel reducible to the isom. of $\mathcal{L}$-structures. (For groups, $\mathcal{L} := (1, \cdot, (\,)^{-1})$, for graphs, $\mathcal{L} := (E)$.)

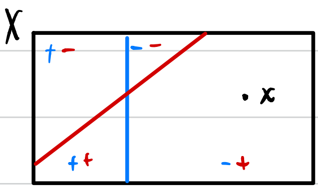
<u>Theorem</u> (Foreman-Weiss). The conjugacy rel. $\sim$ on $\mathrm{Aut}(\mu)$ is not classifiable by ctbl structures.

Proof. Hjorth's turbulence theory...

Entropy

Nevertheless, we will define a Borel function $T \mapsto h(T) : \mathrm{Aut}(\mu) \to [0,\infty]$ that is a $\sim$-invariant (i.e. its constant on each conjugacy class), and hope that maybe on some subset of $\mathrm{Aut}(\mu)$ it's a reduction, i.e. $T \sim S \Leftrightarrow h(T) = h(S)$. This function is **entropy**. Static entropy (without a transformation) was defined and developed by Shannon in the 40's and 50's, creating information theory. The notion of dynamical entropy was developed by Kolmogorov in 1958, who developed it to prove that the Bernoulli shifts $(2^{\mathbb{Z}}, \{\frac{1}{2}, \frac{1}{2}\}^{\mathbb{Z}})$ and $(3^{\mathbb{Z}}, \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}^{\mathbb{Z}})$ are not measure-isomorphic.

To define it for a fixed $T \in \mathrm{Aut}(\mu)$, we first need a motivation from the 20-questions game. In this game, we have a set X of objects. Player 1 chooses $x \in X$,



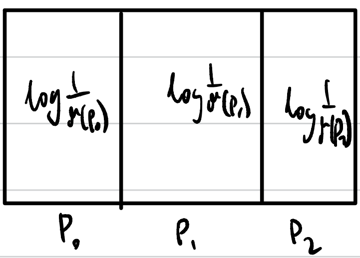
and Player 2 tries to guess x by asking 20 yes/no questions. These 20 questions partition X into 2^{20} pieces and the answers determine a single piece. If that piece has >1 elements, Player 2 is not guaranteed to win. To maximize the chance of winning the best partition into 2^{20} pieces would be the one with pieces having roughly the same size.

For a randomly chosen x, if you learn that x is in a piece of size S, then the information you gain should be inversely proportional to S, so proportional to $\frac{1}{S}$.

Let (X, μ) be a st. prob. space. For a meas. subset $A \subseteq X$, We define $\text{info}_\mu(A)$ as $\log \frac{1}{\mu(A)}$ ('' 0 if $\mu(A)=0$.) The choice of log is because of the following example: let (Y,ν) be a st. prob space, $B \subseteq Y$, and consider $\text{info}_{\mu\times\nu}(A\times B)$. The info function is supposed to measure the information gained after learning that a randomly chosen point is in that set. Since infos for A and for B should add up to $\text{info}(A\times B)$, we put the log, so $\text{info}(A\times B) = \log\frac{1}{\mu(A)\cdot\mu B} = \log\frac{1}{\mu(A)} + \log\frac{1}{\mu(B)}$ $= \text{info}(A) + \text{info}(B)$.

Let's now define the information function $i_{\mathcal{P}}$ for a given ctbl partition $\mathcal{P} = \{P_n\}_{n\in\mathbb{N}}$ of X into measurable pieces P_n:

$$i_{\mathcal{P}} : X \to [0,\infty), \qquad i_{\mathcal{P}} = \sum_{n\in\mathbb{N}} \mathbb{1}_{P_n}\cdot \text{info}(P_n).$$



The (static) entropy of the partition $\mathcal{P}$ is $h(\mathcal{P}) := \mathbb{E}(i_{\mathcal{P}}) = \int_X i_{\mathcal{P}}\, d\mu =$

$$= \sum_{n\in\mathcal{P}} \mu(P_n)\cdot\text{info}(P_n) = -\sum_{n\in\mathcal{P}} \mu(P_n)\log\mu(P_n).$$

To motivate the def. of entropy for a pmp action of $\mathbb{Z}$ on (X,μ), i.e. for $T\in\text{Aut}(\mu)$, let's modify the 20-question game to involve dynamics.

Let $T : X \circlearrowleft X$. Player 2 fixes a finite set of questions in advance, but they're allowed to ask all these questions every day, where think of T as a passage of a unit of time (say a day), so

$$\dots T^{-2}x \quad T^{-1}x \quad \overset{\text{today}}{x} \quad \overset{\text{tomorrow}}{Tx} \quad T^2x \dots$$

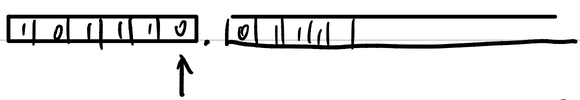
and $x \in X$ is what Player 1 has in mind. Let's give examples where Player 2 has a winning strategy, i.e. there is partition $\mathcal{P}$ of X s.t. no matter what $x \in X$ Player 1 chooses, their x is uniquely determined by its itinerary $(\dots, T^{-2}x, T^{-1}x, x, Tx, T^2x, \dots)$.

Examples. (a) Let $X := \mathbb{R}^+$ and $T(x) := 2x$. I claim that there is 1 question (i.e. partition into 2) s.t. asking that question about

$$\dots \frac{1}{2^2}x \quad \frac{1}{2}x \quad x \quad 2x \quad 2^2x \quad 2^3x \dots$$

determines x uniquely.

Consider the binary rep. of reals. T just shifts it to the left and T^{-1} to the right.



question: is this digit 1?

(b) What about $X := \mathbb{R}$ and $T(x) = x+1$?

This is isom. to (a) by the map $x \mapsto 2^x$.

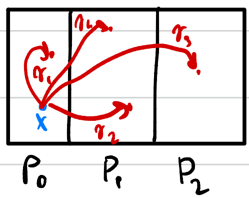
So the pull-back of the winning question in (a) would win here.

Def. For a Borel action of a ctbl group Γ on a st. Borel space X, a Borel partition $\mathcal{P} := \{P_n : n \in \mathbb{N}\}$ is called *generating* if Player 2 wins the game with this partition, i.e. the function:

shift action

$$x \mapsto (\text{the index } n \text{ s.t. } \gamma \cdot x \in P_n)_{\gamma \in \Gamma} : X \to \mathbb{N}^\Gamma$$

is injective.



We refer to this map as the *itinerary map* for $\mathcal{P}$. Note that this map is equivariant.

DST proves that this is equivalent to saying that $\bigcup_{\gamma \in \Gamma} \gamma \cdot \mathcal{P} = \{\gamma \cdot P : \gamma \in \Gamma, P \in \mathcal{P}\}$ generates the whole Borel σ-algebra of X, hence the name.